An Accelerated Procedure for Hypergraph Coarsening on the GPU

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***Abstract*—One of the obstacles in accelerating sparse graph applications using GPUs is load imbalance, which in certain cases causes threads to stall. We investigate a specific application known as hypergraph coarsening and explore a technique for addressing load imbalance. The hypergraph is a generalization of the graph where one edge may connect more than two nodes. Many problems of interest may be expressed in terms of optimal partitioning of hypergraphs where the edge cut is minimized. The most costly step in hypergraph partitioning is hypergraph coarsening, the process of grouping nodes with similar connectivity patterns into one node to yield a new hypergraph with fewer nodes. Hypergraph coarsening proves to be computationally challenging on GPUs because many hypergraphs exhibit an irregular distribution of connections. To address the resulting load imbalance, we explore a novel task allocation scheme to distribute work more evenly among GPU threads.**

I. Introduction

A hypergraph is a generalization of a graph where it replaces edges in a graph with hyperedges that connect multiple vertices. It provides a key modeling flexibility that enables accurate formulation of a wide range of computing problems, from VLSI design [1] to social networks [2] and image classification [3], [4]. With the popularity of hypergraphs in recent years, there has been a strong interest to achieve accurate yet effective partitioning in the hypergraph context.

A *weighted* hypergraph $G = (V, E, w)$ consists of a set of nodes V, a set of hyperedges E, and the weight function $w : E \to [0, +\infty)$ that assigns a weight to each hyperedge. Let $E = \{e_0, \cdots, e_{m-1}\}$ and $V = \{v_0, \cdots, v_{n-1}\}$, where m and n are the number of hyperedges and nodes, respectively, and each $e_i \in E$ is a subset of nodes in V. See an example hypergraph in Figure 1.

A *bipartition* of the hypergraph G comprises a subset of nodes W and its complement $\overline{W}$. An optimal bipartition minimizes the *edge cut* defined by

$$EC(W) = \sum_{\substack{e \cap W \neq \emptyset \\ e \cap \overline{W} \neq \emptyset}} w(e). \tag{1}$$

Here, we impose a *balance constraint* so that W and $\overline{W}$ are comparable in size, that is, the *imbalance coefficient*

$$\epsilon(W) = \frac{2 \cdot \max\{|W|, |\overline{W}|\} - |V|}{|V|} \tag{2}$$

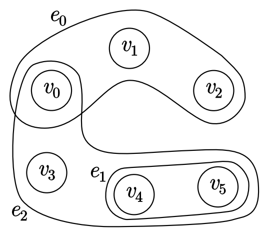


Fig. 1. An example hypergraph with 6 nodes and 3 hyperedges

is made smaller than the prescribed threshold.

The most computationally demanding step in hypergraph partitioning is *hypergraph coarsening*, the process of grouping nodes with similar connectivity patterns into one node to yield a simpler hypergraph. A typical hypergraph partitioner is known to spend up to 91% of its processing time on this step [5].

To this end, in this paper, we focus on the hypergraph bipartitioning problem and present a multi-level hypergraph coarsening framework amenable to GPUs. Unlike parallel algorithms for CPU, we modify the algorithms significantly based on GPU threads, resulting in a parallel Mondriaan algorithm [5], the employment of a family of compiler primitives, as well as a parallel suitor matching algorithm.

Experiments show that our GPU-based implementation outperforms the sequential procedure for coarsening for large hypergraphs. Once a good bipartition of the simpler hypergraph obtained by coarsening, it is then used to approximate a bipartition for the original hypergraph. Although we only consider the case of bipartition in this paper, the problem is readily generalized to multiple categories via recursive bipartition.

II. Hypergraph partitioning and coarsening

The problem of computing an optimal bipartition of a hypergraph is known to be NP-complete [6]. We adopt a class of approximation algorithms called the *multi-level paradigm* [1], [5]. First, a series of increasingly coarse approximations to the full problem is computed until the approximation becomes so small that computing an optimal bipartition is easy. The nodes are combined into clusters based on their connectivity patterns. Second, the solution to the coarsest approximation is projected to the previous approximation. Finally, the projected solution is refined to increase the approximation quality. The

```
Input: A weighted hypergraph $G = (E, V, w)$
Output: A subset $W$ of $V$ and its complement $\overline{W}$
 1: procedure PARTITION($E, V, w$)
 2:     if $|V| < M$ then            ▷ hypergraph is small enough
 3:         return BRUTEFORCE-PARTITION($E, V, w$)
 4:     end if
 5:     $(V', f) \leftarrow$ COARSEN($E, V$)
        ▷ $f$ maps nodes to clusters; $V'$ is a set of clusters
 6:     $(E', w') \leftarrow$ APPLY-COARSENING($f, E, w$)
        ▷ Apply $f$ to obtain approximation $G' = (E', V', w')$
 7:     $W' \leftarrow$ PARTITION($E', V', w$)
 8:     $W \leftarrow$ PROPAGATE($W', f$)
 9:     $W_2 \leftarrow$ REFINE($W, E, V, w$)
10:     return $W_2$
11: end procedure
```

Fig. 2. Multi-level paradigm

second and third steps are repeated until a solution for the full problem is obtained. See Figure 2 for details.

A. Hypergraph coarsening

The central piece of the multi-level paradigm is hypergraph coarsening. We compute clusters of nodes so that the connectivity patterns are preserved. Nodes that belong to a similar list of hyperedges should belong to the same cluster. A new hypergraph is then formed whose nodes are clusters of nodes in the original hypergraph. The new hypergraph thus has fewer nodes than the original hypergraph.

We define *coarsening* of $G = (E, V, w)$ as an order pair (V', f) where V' is a set of clusters and $f : V \rightarrow V'$ is a map that maps nodes to clusters (Line 5 of Figure 2). Later in this section, we will show how to compute a good coarsening map f. Once a coarsening map $f : V \rightarrow V'$ is determined, the corresponding approximation hypergraph $G' = (E', V', w')$ is computed (Line 6). First, we compute the new set of hyperedges $E' = \{e'_0, \cdots, e'_{m-1}\}$ by

$$e'_i = \{v' \in V' : e_i \cap f^{-1}(v') \neq \emptyset\}. \tag{3}$$

In other words, a hyperedge $e'_i \in E'$ contains cluster v' whenever the corresponding hyperedge $e_i \in E$ contains some node belonging to that cluster. Second, we define the new weight function w' by

$$w'(e'_i) = w(e_i). \tag{4}$$

Note that this is well-defined because $|E'| = |E| = m$.

After a partition $(W', \overline{W'})$ is computed for the approximation hypergraph $G' = (E', V', w')$ (Line 7 of Figure 2), it is propagated to the original hypergraph G (Line 8). A partition $(W, \overline{W})$ of G is finally obtained by

$$W = f^{-1}(W'). \tag{5}$$

B. Refinement

After a bipartition $(W, \overline{W})$ of G is obtained, we refine it for quality. The Fiduccia-Mattheyses algorithm [1] is a heuristic algorithm that lowers the edge cut $EC(W)$ by moving nodes into and out of W.

C. Connection to weighted graph matching

It remains to determine a good coarsening map $f : V \rightarrow V'$. To minimize the edge cut $EC(W)$, we form clusters so that nodes in the same cluster belong to a similar list of hyperedges, as nodes are put in the same categories as their clusters.

We first introduce some basic terminology. The node w is said to be a *neighbor* of v whenever there is a hyperedge e containing both. The *similarity* between nodes w and v is defined to be the total weight of all hyperedges that contain both nodes. To compute similarities, we represent $G = (E, V, w)$ as an $m \times n$ sparse matrix $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in e_i \\ 0 & \text{if } v_j \notin e_i. \end{cases} \tag{6}$$

The similarity between nodes v_i and v_j is given by the sum

$$\sum_{v_i, v_j \in e_k} w(e_k) = \sum_{k=0}^{m-1} a_{ki} a_{kj} w(e_k). \tag{7}$$

Here, we use the fact that a_{ki} and a_{kj} are both 1 if and only if both v_i and v_j belong to hyperedge e_k. Define the *weighted dot product* $\langle \cdot, \cdot \rangle_w$ by

$$\langle a_i, a_j \rangle_w = \sum_{k=0}^{m-1} a_{ki} a_{kj} w(e_k) \tag{8}$$

where a_i denotes the i-th column vector of A. Then the similarity between nodes v_i and v_j is simply $\langle a_i, a_j \rangle_w$.

The problem of computing $f : V \rightarrow V'$ is now reduced to the weighted matching problem if f maps at most two nodes to each cluster in V'. Let us define the *metric closure* $M(G) = (E_M, V, \zeta)$ of $G = (E, V, w)$ as follows:

- $M(G)$ is an ordinary graph: $|e| = 2$ for each $e \in E_M$.
- v_i and v_j are connected by an edge $e_{ij} \in E_M$ if and only if they are neighbors.
- If e_{ij} is the edge connecting nodes v_i and v_j, then $\zeta(e_{ij}) = \langle a_i, a_j \rangle_w$.

An optimal weighted matching of $M(G)$ matches each node in V to an adjacent node, so that the total weight of the matched edges is maximized. Since we wish to cluster nodes with a high level of similarities between them, we set $f(v_i) = f(v_j) = \min\{i, j\}$ whenever v_i is matched to v_j under the weighted matching problem. We set $f(v) = v$ if v is left unmatched.

III. RELATED WORK

There has been an extensive research on the subject of weighted matching. For example, several parallel algorithms for the weighted matching problem are described in [7], [8]. Both papers assume that the adjacency lists for the nodes are available for constant-time lookups. This assumption holds for ordinary graphs – it takes a constant time to find all neighbors of a given node. However, the assumption fails to hold for hypergraphs. Since adjacency in hypergraphs is defined only implicitly, to find neighbors of v, we must scan all hyperedges containing v and collect the nodes in those hyperedges. Hence, it takes at least linear time to look up the neighbors. Auer [5] adapted weighted matching algorithms to hypergraph partitioning. A class of approximate matching algorithms [9], [10] is coupled with a class of neighbor-finding algorithms. The adjacency lists are computed as they are needed and the metric closure is only implicitly accessed. Some of the matching and neighbor-finding algorithms are inherently sequential, however.

Çatalyürek et. al. [11] presented two parallel algorithms for hypergraph coarsening on multi-core CPUs. One uses atomic lock operations to prevent inconsistent matching decisions made by multiple threads. Another allows threads perform matchings as they would in a sequential setting and later resolves incompatible matchings later. Both algorithms achieve a linear speedup with respect to the number of physical CPU cores. The algorithms must be modified significantly for GPUs, however, because they operate under the assumption that threads can execute different strings of instructions. This assumption is true for multi-core CPUs but not for GPUs, where hardware threads are not as independent. We shall present a way to overcome this limitation in Section V.

IV. An algorithm for hypergraph coarsening

We first describe a sequential implementation of hypergraph coarsening. The implementation consists of two algorithms: the Mondriaan algorithm and the Greedy matching algorithm.

A. Mondriaan algorithm

The Mondriaan algorithm [5], described in Figure 3, computes the list of all neighbors v_k of a given node v_j and the associated similarities $\langle v_j, v_k \rangle_w$. Rather than computing $\langle v_j, v_k \rangle_w$ for all columns k, we exploit the sparse representation of the matrix A. First, we scan the nonzero entries of the j-th column a_j to construct a list of all hyperedges that contain v_j. Second, for each hyperedge e_i containing v_j, we scan all nonzero entries of e_i to construct a list of all nodes v_k in e_i. If both v_k and v_j are in e_i, then the nodes share one hyperedge and the similarity score $S(k)$ is incremented by $w(e_i)$. We repeat this process for all other hyperedges e_i's, so that each $S(k)$ would eventually contain the total weighted sum of all hyperedges containing both v_k and v_j, namely $\langle v_k, v_j \rangle_w$.

B. Greedy matching algorithm

A greedy approach to matching is outlined in Figure 4. In this algorithm each v_j is matched to the neighbor v_K most similar to it. As soon as v_j and v_K are matched, they are removed from the pool of nodes available for matching.

V. GPU implementation

A. GPU fundamentals

In recent years, GPUs have gained popularity as a general-purpose parallel accelerator. GPUs adopt a many-core architecture where a large number of hardware threads deliver a high instruction throughput. To simplify control logic, GPUs adopt the single-instruction-multiple-data (SIMD) paradigm where a group of hardware threads executes an identical set of instructions on multiple portions of data.

In particular, hardware threads on NVIDIA GPUs are organized in *warps*, or units of 32 threads. Threads in the

```
Input: A sparse matrix $A$ induced by $G = (E, V, w)$; the
    associated weight function $w$; the index $j$ of the node $v_j$
    whose neighbors are to be computed; and an array mate
    whose $k$-th entry gives the index of the node to which $v_k$
    is currently matched ($\mathtt{mate}(k) = -1$ if unmatched yet).
Output: A set $V$ of unmatched neighbors of node $v_j$; and an
    array $S$ where $S(k) = \langle a_j, a_k \rangle_w$
 1: procedure MONDRIAAN($A, w, j, \mathtt{mate}$)
 2:     $S(k) \leftarrow 0$ for all $0 \leq k < n$
 3:     for each row index $i$ for which $a_{ij} = 1$ do
 4:         for each column index $k$ for which $a_{ik} = 1$ do
 5:             if $\mathtt{mate}(k) = -1$ and $k \neq j$ then
                ▷ node $v_k$ has not been matched yet
                  and shares hyperedge $e_i$ with node $v_j$
 6:                 if $S(k) = 0$ then
 7:                     $V \leftarrow V \cup \{k\}$
 8:                 end if
 9:                 $S(k) \leftarrow S(k) + w(e_i)$
10:             end if
11:         end for
12:     end for
13:     return $(V, S)$
14: end procedure
```

Fig. 3. The sequential Mondriaan neighbor-finding algorithm

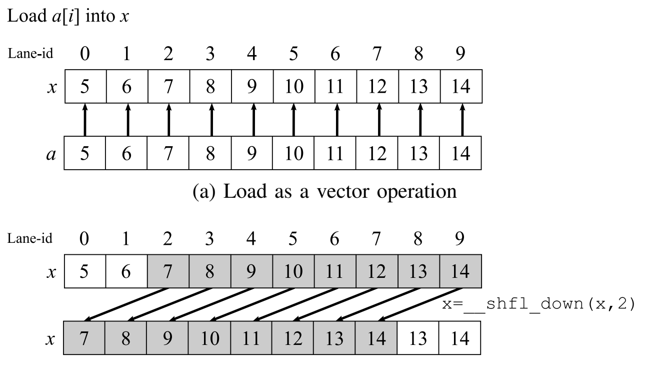
```
Input: A sparse matrix $A$ induced by $G = (E, V, w)$; and the
    associated weight function $w$
Output: An array mate whose $k$-th entry is the index of the
    node $v_k$ is being paired with ($\mathtt{mate}(k) = -1$ if unmatched)
 1: procedure COMPUTE-MATCHING($A, w$)
 2:     $\mathtt{mate}(j) \leftarrow -1$ for all $0 \leq j < n$
 3:     for $j$ from $0$ to $n - 1$ do
 4:         if $\mathtt{mate}(j) = -1$ then
 5:             $(V, S) \leftarrow$MONDRIAAN($A, w, j, \mathtt{mate}$)
 6:             $K \leftarrow \arg\max_k S(k)$
                ▷ $v_K$ is the neighbor most similar to $v_j$
 7:             $\mathtt{mate}(j) \leftarrow \min\{j, K\}$
 8:             $\mathtt{mate}(K) \leftarrow \min\{j, K\}$
                ▷ match nodes $v_j$ and $v_K$
 9:         end if
10:     end for
11:     return mate
12: end procedure
```

Fig. 4. The sequential greedy matching algorithm

same warp share a single instruction counter. If all threads in a warp execute an identical instruction on adjacent data elements, one single vector operation is loaded to the shared instruction counter. On the other hand, if only some threads in a warp execute a given instruction depending a specified condition, the corresponding operation is still loaded to the shared instruction counter but the remaining threads stall. The inactive threads become active once the instruction counter receives a new operation that applies to those threads. Consequently, in cases where a few threads receive a lion's share of workload, the other threads in the same warp remain stalled much of time. This phenomenon is known as *warp divergence* and has a debilitating effect on performance. A detailed coverage of

```
Input: A sparse matrix $A$ induced by $G = (E, V, w)$; the
    associated weight function $w$; the index $j$ of the node $v_j$
    whose neighbors are to be found; a function object $F$, to
    be called with the index of each neighbor of $v_j$
 1: procedure MONDRIAAN-PARALLEL($A, w, j, F$)
 2:     for each row index $i$ for which $a_{ij} = 1$ in parallel do
        ▷ distribute work across warp
 3:         $\omega \leftarrow 32$                ▷ number of threads per warp
 4:         $l \leftarrow$ the lane-id of the current thread ($0 \le l < \omega$)
 5:         $a \leftarrow$ nnz($e_i$)              ▷ # of nonzero entries
 6:         $b \leftarrow$INCLUSIVE-SUM-SCAN-WARP($a$)
 7:         $S \leftarrow$SHFL($b$, $\omega - 1$)
            ▷ # of nonzero entries assigned to current warp
 8:         $b' \leftarrow b - a$              ▷ convert to exclusive scan
 9:         $L \leftarrow \lceil S/\omega \rceil (l + 1)$
10:         for $z$ from 0 to $\omega - 1$ do
11:             if SHFL($b, z$) $< L$ then
12:                 $E_r \leftarrow z$
13:             end if
14:         end for
15:         $E_c \leftarrow L-$SHFL($b', E_r$)
16:         $S_r \leftarrow$SHFL-UP($E_r$, 1)
17:         $S_c \leftarrow$SHFL-UP($E_c$, 1)
18:         if $l = 0$ then
19:             $S_r \leftarrow 0$
20:             $S_c \leftarrow 0$
21:         end if
22:         $S_r \leftarrow S_r+$ [thread-id of first thread in this warp]
23:         $S_c \leftarrow S_c+$ [thread-id of first thread in this warp]
            ▷ Now $(S_r, S_c)$–$(E_r, E_c)$ is the range of indices
              for nonzero entries assigned to current thread
24:         for each nonzero $a_{ik}$ assigned to current thread do
25:             Call F($k$,$w(e_i)$) ▷ pass weight of hyperedge $e_i$
26:         end for
27:     end for
28: end procedure
```

Fig. 5. The parallel Mondriaan algorithm with collaborative task planning



(a) Load as a vector operation

(b) `__shfl_down` primitive

Fig. 6. Local registers of a warp as a block of memory

warp divergence can be found in [12].

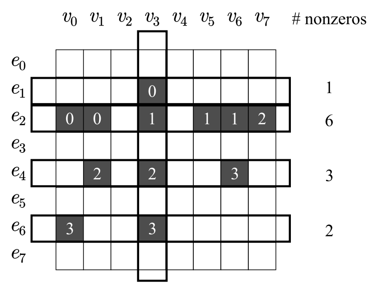
B. Parallel Mondriaan algorithm

We now describe a parallel version of the Mondriaan algorithm. We begin with assigning one nonzero row to each worker thread. Using the number of nonzero entries as a proxy for workload, we see that nonzero rows carry wildly varying amount of work. To eliminate warp divergence that would result otherwise, we employ a collaborative planning to even out workload among the 32 threads in each warp. First, the 32 threads collect all nonzero entries in the 32 nonzero rows assigned to them. The threads then take equal shares of the nonzero entries. Each thread carries a range of 2D indices indicating the set of nonzero entries that it must process. See Figure 5 for details.

For the collaborative planning step, we employ a family of compiler primitives known as Shuffle [13], which is implemented in the current version of the CUDA toolkit [14]. Since NVIDIA GPUs operate under the SIMD paradigm, it is useful to think of local registers as a block of memory cells rather than an isolated box. A single instruction is essentially a vector operation acting on a block of cells (see Figure 6a). The Shuffle primitives let the programmer shuffle the contents of the block (see Figure 6b). In addition to providing inter-warp communication, the Shuffle primitives also act as a lightweight, fine-grained synchronization mechanism — an operation is applied on a group of 32 threads in a synchronous fashion with a single hardware instruction. In addition, common parallel primitives such as reduction and prefix sum are readily implemented using Shuffle [15].

Consider a simple example in Figure 7. In this example, we wish to find neighbors of node 3. (For space consideration, let the warp carry 4 threads instead of the usual 32.) Taking advantage of sparsity, we only consider hyperedges 1, 2, 4, and 6. To distribute workload, we collect all nonzero entries in those hyperedges and divide them into four equal portions. Since hyperedges 1, 2, 4, and 6 contain 12 nonzero entries combined, each thread should get three nonzero entries.

The first step is to compute a fair share of nonzero entries for each thread. We treat local variables as arrays indexed by lane ID. Let $a(l)$ be the number of nonzero entries in the l-th row, and let $b(l)$ be the number of all nonzero entries combined in the first l rows (Lines 5-6 of Figure 5). Then $b(3)$ gives the total number of nonzero entries in the four rows, so set S to



v_0 v_1 v_2 v_3 v_4 v_5 v_6 v_7
# nonzeros
e_0 e_1 e_2 e_3 e_4 e_5 e_6 e_7
1
6
3
2

Fig. 7. The matrix representing a simple hypergraph with 8 nodes and 8 hyperedges. The hyperedges containing node 3 are highlighted. The numbers in the cells indicate the thread to which each entry is assigned.

TABLE I. LOCAL VARIABLES USED IN THE EXAMPLE

Lane ID	0	1	2	3	Lane ID	0	1	2	3
i	1	2	4	6	E_r	1	1	2	3
a	1	6	3	2	SHFL(b', E_r)	1	1	7	10
b	1	7	10	12	E_c	2	5	2	2
S	12	12	12	12	S_r	0	1	1	2
b'	0	1	7	10	S_c	0	2	5	2
$\lceil S/\omega \rceil$	3	3	3	3					
L	3	6	9	12					

be that value (Line 7). Then the fair share is given by $\lceil S/\omega \rceil$. The array $L(\cdot)$ in Line 9 has a special meaning: number all 12 nonzero entries from 0 to 11, and $L(l)$ would be one past the index of the last entry being assigned to thread l. For instance, thread 0 would receive entries 0 to 2, thread 1 would receive entries 3 to 5 and so forth.

Once we find the range of entries being assigned to each thread, it only remains to convert the range into a pair of 2D coordinates in the matrix. For example, we would like to locate the row and column containing entry 3. The loop in Lines 10-14 sets $E_r(l)$ to be the largest index for which $b'(E_r(l)) < L(l)$. The effect is that $E_r(l)$ gets the row index of entry $L(l)$. So $E_r(0) = 1$ indicates that entry 3 is in row 1. Now that the row indices for ranges are known, let us find the corresponding column indices. Looking at the exclusive sum scan, we realize that $b'(E_r(l))$ gives the number of nonzeros in the rows preceding row $E_r(l)$, namely rows 0 to $E_r(l)-1$. Hence, subtracting $b'(E_r(l))$ from $L(l)$ gives the desired column index for entry $L(l)$ (Line 15). For instance, entry 3 is in column 2. Once (E_r, E_c) is found, shift the arrays to the right by one to obtain (S_r, S_c) (Lines 16-21).

C. Parallel suitor-matching algorithm

To divide the matching task into parallel pieces, matching decisions should be made by multiple threads at once. Hence, we cannot rely on node removal to prevent conflicting decisions, as described in Figure 4. The parallel algorithms discussed in Section III adopt the suitor approach, where the nodes make preliminary matching decisions, and only compatible decisions are made final. More specifically, each node v makes a tentative proposal to pair with the node w that is most similar to v. If w also makes the counter-proposal to pair with v, the proposals are made final, and v and w are matched. We adopt the queue-based suitor-matching algorithm presented in [7]: whenever v is matched with another node, we enqueue v to the queue Q_C. After the first round of compatible proposals are processed, the neighbors of nodes in Q_C will be considered for additional matching. See Figure 9 for details.

In the second phase of the algorithm, we make a list of all neighbors for the nodes in Q_C. We recompute matching proposals for those neighbors that tried to pair with nodes in Q_C but did not succeed. Since the best candidates have already been matched away, the neighbors will be matched with their second-best candidates. From the matching proposals, collect all compatible ones and make them final. Enqueue to Q_C all newly matched nodes so that their unmatched neighbors would be considered in the following round. Repeat the second phase of the algorithm until there is no more matching to be made, i.e. until Q_C becomes empty.

VI. Experimental results

We tested our GPU implementation with sparse matrices from [16]. All experiments were done on a dual 2.0 GHz Intel® Xeon® E5-2620 CPU and four NVIDIA® Tesla® K20c GPUs with CUDA 6.5. Each GPU has 2,496 cores divided into 13 streaming multiprocessors and a total memory of 5 GB. We compare the performance of our GPU implementation with the sequential implementation of Mondriaan neighbor-finding algorithm [17]. Other sequential neighbor-finding algorithms offer better performance but lack parallel counterparts.

TABLE II. Distribution of nonzero cardinality of columns

Input	Min	Q1	Median	Q3	Max	Mean
`flickr`	1	1	1	4	8549	12
`wikipedia`	0	0	2	6	75547	12
`stanford`	0	2	5	9	255	8
`stanford-berkeley`	0	3	6	12	249	11

TABLE III. Speedup over sequential coarsening algorithm [17]

Input	GPU (s)	Sequential (s)	Speedup
`flickr`	58.69	876.51	14.93
`wikipedia`	50.42	810.60	16.08
`stanford`	145.54	65.06	0.45
`stanford-berkeley`	625.60	40.79	0.07

We observe in Table III that hypergraphs with a long-tailed distribution of nonzero entries exhibit good speedups. One of the major factors is the number of nonzero entries in each column, i.e. the number of hyperedges that contain each node. In Table II, we see that for all four data sets, over 90% of the columns have fewer than 32 nonzero entries, i.e. most nodes belong to fewer than 32 hyperedges. The difference between the first two sets and the last two is the presence of heavily connected nodes. The sets `flickr` and `wikipedia` contain so many outliers that the arithmetic mean is far larger than the median. On the other hand, the sets `stanford` and `stanford-berkeley` do not contain as many outliers. Unfortunately, our parallel implementation exhibits a good speedup for the first two sets but not for the last two.

To verify our speculation, we generated a synthetic hypergraph with an extreme long-tailed distribution of nonzero entries. Let A be a 700,000 $\times$ 700,000 sparse matrix with zeros everywhere except for the first 1,000 columns. Each of the 1,000 columns have exactly 100,000 entries of 1 randomly distributed in the column and zeros everywhere else. For this matrix, our GPU implementation computed the coarsening map in 262 seconds, whereas the sequential reference in 32,240 seconds!

VII. Connection to image classification

Recent work in computer vision [3], [4] casts the problem of unsupervised image classification as that of hypergraph partitioning. The idea is that hyperedges capture higher-order relationships among sample images that pairwise connections do not.

We outline the steps of obtaining a weighted hypergraph from a given set of images to be classified. For simplicity, we assume two categories of images.

1) Extract local features of sample images using transformation-invariant descriptors such as SIFT [18].
2) Pool local features into one summary vector for each image. The Locality-constrained Linear Coding (LLC) algorithm [19] first converts local features into short codes indicating their positions relative to one another. The codes are then pooled and concatenated to produce one final summary vector for each picture. The process is designed such that information relating the dominant object in the picture is well represented while extraneous backgrounds and noise are pruned.

```
procedure LAMBDA1($k, \texttt{inc}; j, S, \texttt{mate}, K$)
    $l \leftarrow$[thread-id of current thread]
    $\texttt{max\_score} \leftarrow 0$
    $\texttt{max\_id} \leftarrow -1$
    if $\texttt{mate}(k) = -1$ then
        ATOMICADD($S(k), \texttt{inc}$)
        if $S(k) > \texttt{max\_score}$ then
            $\texttt{max\_score} \leftarrow S(k)$
            $\texttt{max\_id} \leftarrow k$
        end if
    end if
    $K \leftarrow \texttt{max\_id}$
end procedure

procedure LAMBDA2($k, \cdot; j, S, Q_I, \texttt{mate}$)
    if $k \neq \texttt{mate}(j)$ and $\texttt{cand}(k) = j$
     and ATOMICCAS($S(k), 0, 1$) $= 0$ then
        Enqueue $k$ to $Q_I$
    end if
end procedure
```

Fig. 8. Helper lambdas (templates for function objects)

3) Compute pairwise distance between nodes (input pictures) using the associated summary vectors.
4) For each n-th sample, collect k nearest neighbors and put them in the n-th hyperedge.
5) Assign weights to hypergraphs to reflect the compactness of member nodes [20].
6) Compute a bipartition for the resulting hypergraph.

VIII. Conclusion

This paper presented an accelerated procedure for hypergraph coarsening. A novel task planning scheme was proposed to boost performance on NVIDIA GPUs, where instruction counters are shared by multiple hardware threads. Our GPU implementation outperformed a comparable sequential implementation for hypergraphs which contain heavily-connected nodes. However, when nodes are evenly connected, our implementation did not perform as well as expected.

We ascribe the discrepancy to static task allocation: Each column of A is assigned at least 32 threads. For columns with many nonzero entries, the threads are fully utilized, while for columns with nonzero entries fewer than 32, the threads are under-utilized and thus stall for the lack of work. The problem occurs because each thread is statically assigned a single column. Assigning multiple columns to GPU threads should address this issue. To that end, we are investigating a more general task allocation strategy which will lead to a higher level of thread utilization.

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```
Input: A sparse matrix $A$ induced by $G = (E, V, w)$; and the
    associated weight function $w$
Output: An array mate whose $k$-th entry is the index of the
    node $v_k$ is being paired with ($\texttt{mate}(k) = -1$ if unmatched)
procedure COMPUTE-MATCHING-PARALLEL($A, w$)
    $\texttt{mate}(j) \leftarrow -1$ for all $0 \leq j < n$
    $\texttt{cand}(j) \leftarrow -1$ for all $0 \leq j < n$
    ▷ array to store tentative matching decisions
    ▷ Phase 1
    for $j$ from $0$ to $n-1$ do
        $F_1 \leftarrow$ LAMBDA1($\cdot, \cdot; j, S, \texttt{mate}, K$)
        MONDRIAAN-PARALLEL($A, j, F_1$)
        ▷ after this call, $S(k) = \langle a_j, a_k \rangle_w$ and
          $K(l) = \arg\max_k S(k)$ for thread $l$
        Perform reduction on array $K(\cdot)$ to obtain
          $K = \arg\max_k S(k)$
        $\texttt{cand}(j) \leftarrow K$
    end for
    for $j$ from $0$ to $n-1$ in parallel do
        if $\texttt{cand}(j) = \texttt{cand}(\texttt{cand}(j))$ then
        ▷ compatible proposals – match them
            $\texttt{mate}(j) \leftarrow \texttt{cand}(j)$
            Enqueue $j$ to queue $Q_C$
        end if
    end for
    ▷ Phase 2
    while $Q_c$ is not empty do
        for $j$ in $Q_C$ do
            $F_2 \leftarrow$ LAMBDA2($\cdot, \cdot; j, S, Q_I, \texttt{mate}, \texttt{cand}$)
            MONDRIAAN-PARALLEL($A, j, F_2$)
            ▷ after this call, $Q_I$ contains indices of nodes
              that proposed to pair with $v_j$ but did not
              succeed; and $S(k)$ indicates whether $k \in Q_I$
        end for
        for $j$ in $Q_I$ do
            $F_3 \leftarrow$ LAMBDA1($\cdot, \cdot; j, S, \texttt{mate}, K$)
            MONDRIAAN-PARALLEL($A, j, F_3$)
            ▷ after this call, $S(k) = \langle a_j, a_k \rangle_w$
              for node $v_k$ available for matching and
              $K(l) = \arg\max_k S(k)$ for thread $l$
            Perform reduction on array $K(\cdot)$ to obtain
              $K = \arg\max_k S(k)$
            $\texttt{cand}(j) \leftarrow K$
        end for
        for $j$ from $0$ to $n-1$ in parallel do
            if $\texttt{cand}(j) = \texttt{cand}(\texttt{cand}(j))$ then
            ▷ compatible proposals – match them
                $\texttt{mate}(j) \leftarrow \texttt{cand}(j)$
                Enqueue $j$ to queue $Q_N$
            end if
        end for
        $Q_C \leftarrow Q_N$
        $Q_N \leftarrow$empty queue
        $Q_I \leftarrow$empty queue
    end while
    return mate
end procedure
```

Fig. 9. The parallel suitor matching algorithm

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